Hodge theory on generalized normal crossing varieties

Dedicated to Slava Shokurov for his sixtieth birthday

Yujiro Kawamata

January 18, 2013

Abstract

We generalize some results in Hodge theory to generalized normal crossing varieties.

1 Introduction

A normal crossing variety is defined to be a variety which is locally isomorphic to a normal crossing divisor in a smooth variety. Similarly we define a generalized normal crossing variety as a variety which is locally isomorphic to a direct product of normal crossing divisors in smooth varieties. Thus a generalized normal crossing variety is a generalized normal crossing divisor in a generalized normal crossing variety of codimension one less. For example, a fiber of a semistable family is a normal crossing variety, and a fiber of a semistable family of Abramovich and Karu [1] is a generalized normal crossing variety. If each irreducible component of the intersections of some of the irreducible components are smooth, then it is called a generalized simple normal crossing (GSNC) variety.

The purpose of this paper is to prove that some Hodge theoretic statements for smooth varieties can be generalized to generalized simple normal crossing varieties. It is similar to the extensions in [10] on simple normal crossing varieties.

As an application, we modify the contents of [9] §4, and correct an error in the proof of [9] Theorem 4.3. Fujino [6] found a simpler alternative proof

of its main theorem, but our construction which shows that the original proof works might also be useful sometime.

In §1, we give definitions concerning the GSNC varieties and GSNC pairs. We prove some lemmas on desingularizations and coverings. In §2, we construct a cohomological mixed Hodge $\mathbf{Q}$-complex on GSNC pairs. The results in §2 are generalized to relative settings in §3, and we prove decomposition theorems when the restricted morphisms from all the closed strata are smooth. We extend the results to the case where there are degenerate fibers in §4 by using the theory of canonical extensions. We prove a vanishing theorem of Kollár type for GSNC varieties in §5 following the original proof of [12]. It is generalized to the $\mathbf{Q}$-divisor version by using the covering lemma, thereby finishing the modification of the argument in [9].

We work over the base field $\mathbf{C}$.

2 GSNC pairs

A reduced complex analytic space X is said to be a *generalized simple normal crossing variety (GSNC variety)* if the following conditions are satisfied:

1. (local) At each point $x \in X$, there exists a complex analytic neighborhood which is isomorphic to a direct product to normal crossing varieties, i.e. varieties isomorphic to normal crossing divisors on smooth varieties.

2. (global) Any irreducible component of the intersection of some of the irreducible components of X is smooth.

The first condition can be put as follows: there exists a complex analytic neighborhood X_x at each point x which is embedded into a smooth variety V with a coordinate system such that X_x is a complete intersection of divisors defined by monomials of coordinates. The *level* of X at x is the smallest number of such equations. For example, a fiber of a semistable family ([1]) satisfies the first condition. X is locally complete intersection, hence Gorenstein and locally equidimensional.

A *closed stratum* of X is an irreducible component of the intersection of some of the irreducible components of X. A closed stratum is smooth by assumption. In the case where X is connected, let $X^{[n]}$ for an integer n denote the disjoint union of all the closed strata of codimension n in X.

The combinatorics of the closed strata is described by the *dual graph*. It is a cell complex where a closed stratum of codimension n corresponds to an n-cell. Any cell is a direct product of simplices due to the local structure of X.

Proposition 1. *There is a Mayer-Vietoris exact sequence*

$$0 \to \mathbf{Q}_X \to \mathbf{Q}_{X^{[0]}} \to \mathbf{Q}_{X^{[1]}} \to \cdots \to \mathbf{Q}_{X^{[N]}} \to 0$$

where $N = \dim X$ and the arrows are alternating sums of restriction homomorphisms.

Proof. We assign fixed orientations to the cells of the dual graph. Then the corresponding chain complex has boundary maps as alternating sums. The sign convention of the sequence of the constant sheaves is defined accordingly. Since each cell is contractible, the sequence is locally exact. $\square$

A Cartier divisor D on X is said to be *permissible* if it does not contain any stratum. We denote by $D|_Y$ the induced Cartier divisor on a closed stratum Y. A *generalized simple normal crossing divisor (GSNC divisor)* B is a permissible Cartier divisor such that $B|_Y$ is a reduced simple normal crossing divisor for each closed stratum Y. In this case B is again a GSNC variety of codimension 1 in X. Namely a GSNC divisor on a GSNC variety of level r is a GSNC variety of level $r+1$. Such an inductive structure may be an advantage. If D is a permissible $\mathbf{Q}$-Cartier divisor whose support is a GSNC divisor, then the round up $\ulcorner D \urcorner$ is well defined. We have $\ulcorner D \urcorner|_Y = \ulcorner D|_Y \urcorner$ for any closed stratum Y.

A *generalized simple normal crossing pair (GSNC pair)* (X, B) consists of a GSNC variety X and a GSNC divisor B on it. For each closed stratum Y of X, we denote $B_Y = B|_Y$. We denote by $B_{X^{[n]}}$ the union of all the simple normal crossing divisors B_Y for irreducible components Y of $X^{[n]}$. A *closed stratum* of the pair (X, B) is an irreducible component of the intersection of some of the irreducible components of X and B.

A morphism $f : X' \to X$ between two GSNC varieties is said to be a *permissible birational morphism* if it induces a bijection between the sets of closed strata of X and X', and birational morphisms between closed strata. A smooth subvariety C of X is said to be a *permissible center* for a GSNC pair (X, B) if the following conditions are satisfied:

1. C does not contain any closed stratum, and the scheme theoretic intersections C_Y of C and the closed strata Y are smooth.

2. At each point $x \in C_Y$, there exists a coordinate system in a neighborhood of x such that B_Y and C_Y are defined by ideals generated by monomials.

A blowing up $f : X' \to X$ whose center is permissible with respect to a GSNC pair (X, B) is called a *permissible blowing up*. It is a permissible birational morphism from another GSNC variety, and the union of the strict transform of B and the exceptional divisor is again a GSNC divisor.

The following is a corollary of a theorem of Hironaka [7].

Lemma 2. *Let (X, B) be a GSNC pair. Consider one of the following situations:*

(a) Let $f : X \dashrightarrow S$ be a rational map to another variety whose domain of definition has non-empty intersection with each closed stratum of the pair (X, B).

(b) Let Z be a closed subset which does not contain any closed stratum of the pair (X, B).

Then there exists a tower of permissible blowing-ups

$$g : (X_n, B_n) \to (X_{n-1}, B_{n-1}) \to \cdots \to (X_0, B_0) = (X, B)$$

where B_{i+1} is the union of the strict transform of B_i and the exceptional divisor, satisfying the following:

(a) There is a morphism $h : X_n \to S$ such that $h = f \circ g$.

(b) The union $\bar{B}_n = B_n \cup g^{-1}(Z)$ is a GSNC divisor on X_n.

Proof. (a) We reduce inductively the indeterminacy locus of the rational map f. For an irreducible component Y of X, the restriction of f to Y is resolved by a tower of permissible blowing-ups of the pair (Y, B_Y) by the theorem of Hironaka. Since the indeterminacy locus of f does not contain any closed stratum, the centers of the blowing-ups do not contain any closed stratum either. Moreover this property is preserved in the process. Therefore the same centers determine a tower of permissible blowing-ups of (X, B). If we do the same process for all the irreducible components of X, then we obtain the assertion.

(b) is similar to (a). We resolve Z at each irreducible component Y of X, since Z does not contain any closed stratum of the pair (Y, B_Y). □

We generalize a covering lemma [8]:

Lemma 3. *Let (X, B) be a quasi-projective GSNC pair. Let D_j ($j = 1, \dots, r$) be permissible reduced Cartier divisors which satisfy the following conditions:*

1. *$D_j \subset B$.*
2. *D_j and $D_{j'}$ have no common irreducble components if $j \ne j'$.*
3. *$D_j \cap Y$ are smooth for all D_j and closed strata Y of X.*

Let d_j be rational numbers, and $D = \sum d_j D_j$. Then there exists a finite surjective morphism from another GSNC pair $\pi : (X', B') \to (X, B)$ such that $B' = \pi^{-1}(B)$ and $\pi^* D$ has integral coefficients.

Proof. The proof is similar to [8] Theorem 17, where the D_j play the role of the irreducible components of D. □

3 Hodge theory on a GSNC pair

We construct explicitly the mixed Hodge structures of Deligne [3] for GSNC pairs.

Let (X, B) be a GSNC pair. We do not assume that X is compact at the moment. We define a cohomological mixed Hodge complex on X when X is projective. We can extend the construction in [10] to a GSNC pair, but we use here the De Rham complex of Du Bois. The underlying local system is the constant sheaf $\mathbf{Q}$ in our case, while it is different and more difficult in the former case.

We define a *De Rham complex* $\bar{\Omega}^\bullet_X(\log B)$ by the following Mayer-Vietoris exact sequence:

$$\begin{aligned} &0 \to \bar{\Omega}^\bullet_X(\log B) \to \Omega^\bullet_{X^{[0]}}(\log B_{X^{[0]}}) \to \Omega^\bullet_{X^{[1]}}(\log B_{X^{[1]}}) \to \\ &\cdots \to \Omega^\bullet_{X^{[N]}}(\log B_{X^{[N]}}) \to 0 \end{aligned}$$

where $N = \dim X$ and the arrows are the alternating sums of the restriction homomorphisms. We have $\bar{\Omega}^0_X \cong \mathcal{O}_X$, and

$$Ri_* \mathbf{C}_{X \setminus B} \cong \bar{\Omega}^\bullet_X(\log B)$$

for the open immersion $i : X \setminus B \to X$.

We define a *weight filtration* on the complex $\bar{\Omega}^\bullet_X(\log B)$ by

$$0 \to W_q(\bar{\Omega}^\bullet_X(\log B)) \to W_q(\Omega^\bullet_{X^{[0]}}(\log B_{X^{[0]}})) \to W_{q+1}(\Omega^\bullet_{X^{[1]}}(\log B_{X^{[1]}})) \to \cdots \to W_{q+N}(\Omega^\bullet_{X^{[N]}}(\log B_{X^{[N]}})) \to 0$$

where the W's from the second terms denote the filtration with respect to the order of log poles.

We define a *weight filtration* $W_q(Ri_*\mathbf{Q}_{X\setminus B})$ as a convolution of the following complex of objects

$$\tau_{\le q}(R(i_0)_*\mathbf{Q}_{X^{[0]}\setminus B_{X^{[0]}}}) \to \tau_{\le q+1}(R(i_1)_*\mathbf{Q}_{X^{[1]}\setminus B_{X^{[1]}}}) \to \cdots \to \tau_{\le q+N}(R(i_N)_*\mathbf{Q}_{X^{[N]}\setminus B_{X^{[N]}}})$$

where τ denotes the canonical filtration and $i_p : X^{[p]} \setminus B_{X^{[p]}} \to X^{[p]}$ are open immersions, such that there are isomorphisms

$$W_q(Ri_*\mathbf{Q}_{X\setminus B}) \otimes \mathbf{C} \cong W_q(\bar{\Omega}^\bullet_X(\log B))$$

for all q as in [10].

We define a *Hodge filtration* as a stupid filtration:

$$F^p(\bar{\Omega}^\bullet_X(\log B)) = \bar{\Omega}^{\ge p}_X(\log B).$$

Then we have the following whose proof can be compared to [10] Lemma 3.2:

Lemma 4.

$$Gr^W_q(Ri_*\mathbf{Q}_{X\setminus B}) \cong \bigoplus_{p\ge 0, dim X^{[p]}-\dim Y = p+q} \mathbf{Q}_Y[-2p-q]$$

$$Gr^W_q(\bar{\Omega}^\bullet_X(\log B)) \cong \bigoplus_{p\ge 0, dim X^{[p]}-\dim Y = p+q} \Omega^\bullet_Y[-2p-q]$$

$$F^r(Gr^W_q(\bar{\Omega}^\bullet_X(\log B))) \cong \bigoplus_{p\ge 0, dim X^{[p]}-\dim Y = p+q} \Omega^{\ge r-p-q}_Y[-2p-q]$$

where the p run all non-negative integers and the Y run all the closed strata of the pair $(X^{[p]}, B_{X^{[p]}})$ of codimension $p+q$.

Proof. We have

$$\begin{aligned}
&\mathrm{Gr}_q^W(Ri_*\mathbf{Q}_{X\setminus B}) \cong \bigoplus_p R^{p+q}(i_p)_*\mathbf{Q}_{X^{[p]}\setminus B_{X^{[p]}}}[-2p-q]\\
&\cong \bigoplus_{p\geq 0, dim X^{[p]}-\dim Y=p+q} \mathbf{Q}_Y[-2p-q]\\
&\mathrm{Gr}_q^W(\bar{\Omega}_X^\bullet(\log B)) \cong \bigoplus_p \mathrm{Gr}_{p+q}^W(\Omega_{X^{[p]}}^\bullet(\log B_{X^{[p]}}))[-p]\\
&\cong \bigoplus_{p\geq 0, dim X^{[p]}-\dim Y=p+q} \Omega_Y^\bullet[-2p-q]\\
&F^r(\mathrm{Gr}_q^W(\bar{\Omega}_X^\bullet(\log B))) \cong \bigoplus_p \mathrm{Gr}_{p+q}^W(F^r(\Omega_{X^{[p]}}^\bullet(\log B_{X^{[p]}})))[-p]\\
&\cong \bigoplus_{p\geq 0, dim X^{[p]}-\dim Y=p+q} \Omega_Y^{\geq r-p-q}[-2p-q]
\end{aligned}$$

□

As a corollary, we have the following whose proof is similar to [10] Theorem 3.3:

Theorem 5. *Let (X, B) be a GSNC pair. Assume that X is projective. Then*

$$((Ri_*\mathbf{Q}_{X\setminus B}, W), (\bar{\Omega}_X^\bullet(\log B), W, F))$$

is a cohomological mixed Hodge $\mathbf{Q}$-complex on X.

Proof. If Y is a closed stratum of the pair $(X^{[p]}, B_{X^{[p]}})$ of codimension $p+q$, then

$$(\mathbf{Q}_Y, \Omega_Y^\bullet, F(-p-q))$$

is a cohomological Hodge $\mathbf{Q}$-complex of weight $2(p+q)$, where $F(-p-q)^r = F^{r-p-q}$. Hence

$$(\mathbf{Q}_Y[-2p-q], \Omega_Y^\bullet[-2p-q], F(-p-q)[-2p-q])$$

is a cohomological Hodge $\mathbf{Q}$-complex of weight $2(p+q)-2p-q=q$. □

Corollary 6. *The weight spectral sequence*

$${}_WE_1^{p,q} = H^{p+q}(Gr_{-p}^W(Ri_*\mathbf{Q}_{X\setminus B})) \Rightarrow H^{p+q}(\mathbf{Q}_{X\setminus B})$$

degenerates at E_2, and the Hodge spectral sequence

$$ {}_F E_1^{p,q} = H^q(\bar{\Omega}_X^p(\log B)) \Rightarrow H^{p+q}(\bar{\Omega}_X^\bullet(\log B)) $$

degenerates at E_1.

Proof. This is [4] 8.1.9. □

When $B = 0$, we have more degenerations:

Corollary 7. *The weight spectral sequence of the Hodge pieces*

$$ {}_{W,r} E_1^{p,q} = H^{p+q}(Gr_{-p}^W(\bar{\Omega}_X^r)) \Rightarrow H^{p+q}(\bar{\Omega}_X^r) $$

degenerates at E_2 for all r.

Proof. The differentials ${}_W d_1^{p,q}$ of the weight spectral sequence in the previous corollary preserve the degree of the differentials. Therefore the E_2-degeneration of the third spectral sequence is a consequence of the first two degenerations. □

4 Relativization

We can easily generalize the results in the previous section to the relative setting when all the closed strata are smooth over the base.

We consider the following situation: (X, B) is a pair of a GSNC variety and a GSNC divisor, S is a smooth algebraic variety, which is not necessarily complete, and $f : X \to S$ is a projective surjective morphism. We assume that, for each closed stratum Y of the pair (X, B), the restriction $f|_Y : Y \to S$ is surjective and smooth.

We define a *relative De Rham complex* $\bar{\Omega}_{X/S}^\bullet(\log B)$ by the following exact sequence

$$ \begin{aligned} &0 \to \bar{\Omega}_{X/S}^\bullet(\log B) \to \bar{\Omega}_{X^{[0]}/S}^\bullet(\log B_{X^{[0]}}) \to \bar{\Omega}_{X^{[1]}/S}^\bullet(\log B_{X^{[1]}}) \to \\ &\cdots \to \bar{\Omega}_{X^{[N]}/S}^\bullet(\log B_{X^{[N]}}) \to 0. \end{aligned} $$

In particular we have

$$ \bar{\Omega}_{X/S}^0(\log B) \cong \mathcal{O}_X. $$

A *weight filtration* on the complex $\bar{\Omega}^\bullet_{X/S}(\log B)$ is defined by using the filtration with respect to the order of log poles on the closed strata as in the previous section:

$$\begin{aligned} &0 \to W_q(\bar{\Omega}^\bullet_{X/S}(\log B)) \to W_q(\bar{\Omega}^\bullet_{X^{[0]}/S}(\log B_{X^{[0]}})) \\ &\to W_{q+1}(\bar{\Omega}^\bullet_{X^{[1]}/S}(\log B_{X^{[1]}})) \to \\ &\cdots \to W_{q+N}(\bar{\Omega}^\bullet_{X^{[N]}/S}(\log B_{X^{[N]}}) \to 0. \end{aligned}$$

We define a *Hodge filtration* by

$$F^p(\bar{\Omega}^\bullet_X(\log B)) = \bar{\Omega}^{\geq p}_X(\log B).$$

Lemma 8. *There is an isomorphism*

$$Ri_*\mathbf{C}_{X\setminus B} \otimes f^{-1}\mathcal{O}_S \cong \bar{\Omega}^\bullet_{X/S}(\log B)$$

such that

$$W_q(Ri_*\mathbf{C}_{X\setminus B}) \otimes f^{-1}\mathcal{O}_S \cong W_q(\bar{\Omega}^\bullet_{X/S}(\log B)).$$

We have again:

Lemma 9.

$$\begin{aligned} Gr^W_q(\bar{\Omega}^\bullet_{X/S}(\log B)) &\cong \bigoplus_{p\geq 0, dim X^{[p]} - \dim Y = p+q} \Omega^\bullet_{Y/S}[-2p-q] \\ F^r(Gr^W_q(\bar{\Omega}^\bullet_{X/S}(\log B))) &\cong \bigoplus_{p\geq 0, dim X^{[p]} - \dim Y = p+q} \Omega^{\geq r-p-q}_{Y/S}[-2p-q] \end{aligned}$$

where the p run all non-negative integers and the Y run all the closed strata of the pair $(X^{[p]}, B_{X^{[p]}})$ of codimension $p+q$.

The following theorem and corrollaries are similar to [10] Theorem 4.1 and Corollary 4.2:

Theorem 10.

$$((R^n(f\circ i)_*\mathbf{Q}_{X\setminus B}, W(-n)), (R^n f_*\bar{\Omega}^\bullet_{X/S}(\log B), W(-n), F))$$

is a variation of mixed Hodge $\mathbf{Q}$-structures on S.

Corollary 11. *The weight spectral sequence*

$$ {}_WE_1^{p,q} = R^{p+q} f_* Gr^W_{-p}(Ri_*\mathbf{Q}_{X\setminus B}) \Rightarrow R^{p+q}(f \circ i)_*\mathbf{Q}_{X\setminus B} $$

degenerates at E_2, and the Hodge spectral sequence

$$ {}_FE_1^{p,q} = R^q f_* \bar{\Omega}^p_{X/S}(\log B) \Rightarrow R^{p+q} f_* \bar{\Omega}^\bullet_{X/S}(\log B) $$

degenerates at E_1.

If $B = 0$, then we have again:

Corollary 12. *The weight spectral sequence of the Hodge pieces*

$$ {}_{W,r}E_1^{p,q} = R^{p+q} f_* Gr^W_{-p}(\bar{\Omega}^r_{X/S}) \Rightarrow R^{p+q} f_*(\bar{\Omega}^r_{X/S}) $$

degenerates at E_2 for all r.

As applications, we deduce *decomposition theorems*:

Corollary 13. *Let (X, B) be a GSNC pair, S a smooth algebraic variety, which is not necessarily complete, and let $f : X \to S$ be a projective surjective morphism. Assume that, for each closed stratum Y of the pair (X, B), the restriction $f|_Y : Y \to S$ is surjective and smooth. Then there is an isomorphism*

$$ R(f \circ i)_*\mathbf{Q}_{X\setminus B} \cong \bigoplus_{p \geq 0} R^p(f \circ i)_*\mathbf{Q}_{X\setminus B}[-p]. $$

Proof. The proof is similar to [11] Theorem 28. Here we use the weight filtration with Lemma 4, the decomposition

$$ Rf_*\mathbf{Q}_Y \cong \bigoplus_{p \geq 0} R^p f_*\mathbf{Q}_Y[-p] $$

of Deligne [2] for each closed startum Y, and the E_2-degeneration of the weight spectral sequence (Corollary 11). □

Corollary 14. *Under the same assumptions as in Corollary 13, there are isomorphisms*

$$ Rf_*\bar{\Omega}^r_{X/S} \cong \bigoplus_{p \geq 0} R^p f_*\bar{\Omega}^r_{X/S}[-p] $$

for all r.

Proof. The proof is similar to [11] Theorem 30. Here we use E_1-degeneration of the Hodge spectral sequence as well as E_2-degeneration of the weight spectral sequence (Corollary 12). □

5 Canonical extension

We prove local freeness theorem by using the theory of canonical extensions when the degeneration locus is a simple normal crossing divisor.

Let (S, B_S) be a pair of a smooth projective variety and a simple normal crossing divisor. We denote $S^o = S \setminus B_S$. Let $H_{\mathbf{Q}}$ be a local system on S^o which underlies a variation of mixed Hodge $\mathbf{Q}$-structures. Assuming that all the local monodromies around the branches of B_S are quasi-unipotent, we define the *lower canonical extension* $\tilde{\mathcal{H}}$ of $\mathcal{H} = H_{\mathbf{Q}} \otimes \mathcal{O}_{S^o}$ as follows.

We take an arbitrary point $s \in B_S$ at the boundary. Let $N_i = \log T_i$ be the logarithm of the local monodromies T_i around the branches of B_S around s, and let t_i be the local coordinates corresponding to the branches. Here we select a branch of the logarithmic function so that the eigenvalues of N_i belong to the interval $2\pi\sqrt{-1}(-1, 0]$. Then the lower canonical extension $\tilde{\mathcal{H}}$ is defined as a locally free sheaf on S which is generated by local sections of the form $\tilde{v} = \exp(-\sum_i N_i \log t_i / 2\pi\sqrt{-1})(v)$ near s, where the v are flat sections of $H_{\mathbf{Q}}$. We note that the monodromy actions on v and the logarithmic functions are canceled and the $\tilde{v}$ are single-valued holomorphic sections of $\mathcal{H}$ outside the boundary divisors.

By [15], the Hodge filtration of $\mathcal{H}$ extends to a filtration by locally free subsheaves, which we denote again by F.

Let $f : X \to S$ be a surjective morphism between smooth projective varieties which is smooth over S^o. We denote $X^o = f^{-1}(S^o)$ and $f^o = f|_{X^o}$. Let $H^q_{\mathbf{Q}} = R^q f^o_* \mathbf{Q}_{X^o}$ for an integer q, let $\mathcal{H}^q = H^q_{\mathbf{Q}} \otimes \mathcal{O}_{S^o}$, and let $\tilde{\mathcal{H}}^q$ be the canonical extension. Then by Kollár [13] and Nakayama [14], there is an isomorphism

$$R^q f_* \mathcal{O}_X \cong \mathrm{Gr}^0_F(\tilde{\mathcal{H}}^q)$$

which extends a natural isomorphism

$$R^q f^o_* \mathcal{O}_{X^o} \cong \mathrm{Gr}^0_F(\mathcal{H}^q).$$

The following theorem will be used in the next section:

Theorem 15. *Let (X, B) be a pair of a projective GSNC variety and a GSNC divisor, (S, B_S) a pair of a smooth projective variety and a simple normal crossing divisor, and let $f : X \to S$ be a projective surjective morphism. Assume that, for each closed stratum Y of the pair (X, B), the restriction $f|_Y : Y \to S$ is surjective and smooth over $S^o = S \setminus B_S$. Denote $X^o =$*

$f^{-1}(S^o)$ and $f^o = f|_{X^o}$. For integers q, let $H^q_{\mathbf{Q}} = R^q f^o_* \mathbf{Q}_{X^o}$ be the local system on S^o which underlies a variation of mixed Hodge $\mathbf{Q}$-structures defined in the preceeding section. Let $\mathcal{H}^q = H^q_{\mathbf{Q}} \otimes \mathcal{O}_{S^o}$, and let $\tilde{\mathcal{H}}^q$ be their canonical extensions. Then the following hold:

(1) The weight spectral sequence of a Hodge piece

$$_{W,0}E_1^{p,q} = R^{p+q} f_* Gr^W_{-p}(\mathcal{O}_X) \Rightarrow R^{p+q} f_* \mathcal{O}_X$$

degenerates at E_2.

(2) There are isomorphisms

$$R^q f_* \mathcal{O}_X \cong Gr^0_F(\tilde{\mathcal{H}}^q)$$

which extend natural isomorphisms

$$R^q f_* \mathcal{O}_{X^o} \cong Gr^0_F(\mathcal{H}^q)$$

for all integers q.

(3) $R^q f_* \mathcal{O}_X$ are locally free sheaves.

Proof. We extend the definition of the weight filtration on $\mathcal{O}_{X^o} = \mathrm{Gr}^0_F(\bar{\Omega}^\bullet_{X^o/S^o})$ to $\mathcal{O}_X$ by an exact sequence

$$\begin{aligned} &0 \to W_q(\mathcal{O}_X) \to W_q(\mathcal{O}_{X^{[0]}}) \to W_{q+1}(\mathcal{O}_{X^{[1]}}) \to \\ &\cdots \to W_{q+N}(\mathcal{O}_{X^{[N]}}) \to 0 \end{aligned}$$

where $W_q(\mathcal{O}_{X^{[p]}}) = \mathcal{O}_{X^{[p]}}$ for $q \ge 0$, and $= 0$ otherwise, for any p. By the canonical extension, we extend the E_2-degeneration of the weight spectral sequence from S^o to S as in [10] Theorem 5.1. Then we reduce the assertion to the theorem of Kollár and Nakayama. □

We have the following decomposition theorem:

Corollary 16. *Under the same assumptions as in Theorem 15, assume in addition that the induced morphisms $f|_Y : Y \to S$ for all closed strata Y have well prepared birational models $f'_Y : Y' \to S$ over S with birational morphisms $\mu_Y : Y' \to Y$ in the sense of [11]. Then there is an isomorphism*

$$Rf_* \mathcal{O}_X \cong \bigoplus_{p \ge 0} R^p f_* \mathcal{O}_X[-p].$$

Proof. By [11] Theorem 30, we have

$$R(f'_Y)_*\mathcal{O}_{Y'} \cong \bigoplus_{p\geq 0} R^p(f'_Y)_*\mathcal{O}_{Y'}[-p].$$

Since $R\mu_{Y*}\mathcal{O}_{Y'} \cong \mathcal{O}_Y$, we obtain

$$R(f|_Y)_*\mathcal{O}_Y \cong \bigoplus_{p\geq 0} R^p(f|_Y)_*\mathcal{O}_Y[-p].$$

Then the rest of the proof is similar to [11] Theorem 28 with the aid of the E_2-degeneration of the weight spectral sequence. □

6 Kollár type vanishing

We shall generalize the vanishing theorem of Kollár [12] to GSNC varieties:

Theorem 17. *Let X be a projective GSNC variety, S a normal projective variety, $f : X \to S$ a projective surjective morphism, and let L be a permissible Cartier divisor on X such that $\mathcal{O}_X(m_1L)$ is generated by global sections for a positive integer m_1. Assume that $\mathcal{O}_X(m_2L) \cong f^*\mathcal{O}_Z(L_S)$ for a positive integer m_2 and an ample Cartier divisor L_S on S, and for each closed stratum Y of X, the restricted morphism $f|_Y : Y \to S$ is surjective. Then the following hold:*

(1) Let $s \in H^0(X, \mathcal{O}_X(nL))$ be a non-zero section for some positive integer n such that the corresponding Cartier divisor D is permissible. Then the natural homomorphisms given by s

$$H^p(X, \mathcal{O}_X(K_X + L)) \to H^p(X, \mathcal{O}_X(K_X + L + D))$$

are injective for all p.

(2) $H^q(S, R^pf_*\mathcal{O}_X(K_X + L)) = 0$ for $p \geq 0$ and $q > 0$.

(3) $R^pf_*\mathcal{O}_X(K_X)$ are torsion free for all p.

Proof. We follow closely the proof of [12] Theorem 2.1 and 2.2. We use the same numbering of the steps.

Step 1. We may assume, and will assume from now, that $\mathcal{O}_X(L)$ is generated by global sections.

We achieve this reduction by constructing a Kummer covering and taking the Galois invariant part as in loc. cit.

Step 2. We prove the dual statement of (1) in the case where $n = 1$ and D is generic: Let $s, s' \in H^0(X, L)$ be general members, and let D, D' be the corresponding permissible Cartier divisors. Then the natural homomorphisms given by s

$$H^p(X, \mathcal{O}_X(-D - D')) \to H^p(X, \mathcal{O}_X(-D'))$$

are surjective for all p.

We go into details in this step in order to show how to generalize the argument in loc. cit. to the GSNC case. Since s and s' are general, $D, D', D+D'$ and $D \cap D'$ are also GSNC varieties. We consider the following commutative diagram:

$$\begin{array}{ccccccc}
H^{p-1}(\mathcal{O}_{D+D'}) & \longrightarrow & H^p(\mathcal{O}_X(-D-D')) & \longrightarrow & H^p(\mathcal{O}_X) & \xrightarrow{d_p} & H^p(\mathcal{O}_{D+D'}) \\
{\scriptstyle b'_{p-1}}\downarrow & & \downarrow & & {\scriptstyle =}\downarrow & & {\scriptstyle b'_p}\downarrow \\
H^{p-1}(\mathcal{O}_{D'}) & \longrightarrow & H^p(\mathcal{O}_X(-D')) & \longrightarrow & H^p(\mathcal{O}_X) & \xrightarrow{e'_p} & H^p(\mathcal{O}_{D'})
\end{array}$$

In order to prove our assertion, we shall prove that (a) b'_{p-1} is surjective, and (b) $\mathrm{Ker}(d_p) = \mathrm{Ker}(e'_p)$.

(a) We consider the following Mayer-Vietoris exact sequence

$$H^{p-1}(\mathbf{C}_{D\cap D'}) \xrightarrow{\bar{a}_p} H^p(\mathbf{C}_{D+D'}) \xrightarrow{\bar{b}_p+\bar{b}'_p} H^p(\mathbf{C}_D) \oplus H^p(\mathbf{C}_{D'}) \xrightarrow{\bar{c}_p-\bar{c}'_p} H^p(\mathbf{C}_{D\cap D'})$$

whose Gr_F^0 is

$$H^{p-1}(\mathcal{O}_{D\cap D'}) \xrightarrow{a_p} H^p(\mathcal{O}_{D+D'}) \xrightarrow{b_p+b'_p} H^p(\mathcal{O}_D) \oplus H^p(\mathcal{O}_{D'}) \xrightarrow{c_p-c'_p} H^p(\mathcal{O}_{D\cap D'}).$$

There is a path connecting D and D' in a linear system $|L|$ on X which gives a diffeomorphism of pairs $(D, D \cap D') \to (D', D \cap D')$ fixing $D \cap D'$. Therefore we have an equality $\mathrm{Im}(\bar{c}_p) = \mathrm{Im}(\bar{c}'_p)$, which implies that $\bar{b}'_p$ is surjective. Hence so is b'_p.

(b) It is sufficient to prove that $\mathrm{Im}(d_p) \cap \mathrm{Ker}(b'_p) = 0$. Using a path connecting D and D', we deduce that $\mathrm{Ker}(e_p) = \mathrm{Ker}(e'_p)$ for $e_p : H^p(\mathcal{O}_X) \to H^p(\mathcal{O}_D)$. Thus

$$\mathrm{Im}(d_p) \cap \mathrm{Ker}(b'_p) = \mathrm{Im}(d_p) \cap \mathrm{Ker}(b_p).$$

Therefore it is sufficient to prove that $\mathrm{Im}(a_p) \cap \mathrm{Im}(d_p) = 0$. We shall prove that

$$\mathrm{Im}(a_p) \cap \mathrm{Im}(d_p) \cap W_q(H^p(\mathcal{O}_{D+D'})) = 0$$

by induction on q.

Let

$$a_{p,q} : \mathrm{Gr}_q^W(H^{p-1}(\mathcal{O}_{D\cap D'})) \to \mathrm{Gr}_q^W(H^p(\mathcal{O}_{D+D'}))$$
$$d_{p,q} : \mathrm{Gr}_q^W(H^p(\mathcal{O}_X)) \to \mathrm{Gr}_q^W(H^p(\mathcal{O}_{D+D'})).$$

be the natural homomorphisms. Then it is sufficient to prove that

$$\mathrm{Im}(a_{p,q}) \cap \mathrm{Im}(d_{p,q}) = 0.$$

For $A = X, D \cap D'$ or $D + D'$, we have the following spectral sequences

$${}_{W,A}E_1^{r,s} = H^{r+s}(\mathrm{Gr}_{-r}^W(\mathcal{O}_A)) = \bigoplus_{\dim A - \dim Y = r} H^s(\mathcal{O}_Y) \Rightarrow H^{r+s}(\mathcal{O}_A).$$

Therefore $d_{p,q}$ is induced from the sum of homomorphisms $H^s(\mathcal{O}_Y) \to H^s(\mathcal{O}_{Y'})$ such that $s = q$, $Y \subset X$, $Y' \subset D + D'$, $Y' \subset Y$ and

$$r = \dim X - \dim Y = \dim(D + D') - \dim Y' = p - q$$

hence $Y' = Y \cap D$ or $Y' = Y \cap D'$.

On the other hand, $a_{p,q}$ is induced from the sum of homomorphisms $H^s(\mathcal{O}_{Y''}) \to H^s(\mathcal{O}_{Y'})$ such that $s = q$, $Y'' \subset D \cap D'$, $Y' \subset D + D'$, $Y'' = Y'$ and

$$r = \dim(D \cap D') - \dim Y'' + 1 = \dim(D + D') - \dim Y' = p - q.$$

Therefore there is no closed stratum Y' of $D + D'$ which receives non-trivial images from both Y and Y'', hence we have our assertion.

Step 3. (1) in the case where $n = 2^d - 1$ for a positive integer d and D generic is an immediate corollary of Step 2.

Step 4. Proof of (2) is the same as in loc. cit.

Step 5. Proof of (3).

This is a generalization of Step 2. We use the notation there. D' is again generic but D is special. More precisely, D is special along a fiber $f^{-1}(s)$ over a point $s \in S$ and generic otherwise. Therefore the intersection $D \cap D'$ is still generic, hence a GSNC variety.

Let $\mu : \tilde{X} \to X$ be the blowing-up along the center $D \cap D'$. $\tilde{X}$ is again a GSNC variety. We denote by $\tilde{Y}$ the closed stratum of $\tilde{X}$ above a closed

closed stratum Y of X. Let $g : \tilde{X} \to \mathbf{P}^1$ be the morphism induced from the linear system spanned by D and D'. Let $U \subset \mathbf{P}^1$ be an open dense subset such that the restricted morphisms $g|_{\tilde{Y}} : \tilde{Y} \to \mathbf{P}^1$ are smooth over U for all the closed strata Y.

(a) In order to prove that b'_p is surjective, we need to prove an inclusion $\mathrm{Im}(c'_p) \subset \mathrm{Im}(c_p)$. For this purpose, we shall prove

$$\mathrm{Im}(c'_p) = \mathrm{Im}(c_p \circ e_p).$$

Since $c_p \circ e_p = c'_p \circ e'_p$, the right hand side is contained in the left hand side. The other direction is the essential part. We note that both c'_p and $c_p \circ e_p$ are parts of homomorphisms of mixed Hodge structures.

Let $\tilde{X}_U = g^{-1}(U)$ and $\tilde{Y}_U = \tilde{Y} \cap \tilde{X}_U$. Then natural homomorphisms

$$\begin{aligned} R^p g_* \mathbf{C}_{\tilde{X}_U} &\to H^p(\mathbf{C}_{D \cap D'}) \times U \\ R^p g_* \mathbf{C}_{\tilde{Y}_U} &\to H^p(\mathbf{C}_{Y \cap D \cap D'}) \times U \end{aligned}$$

underlie respectively morphisms of variations of mixed and pure Hodge structures over U, where the targets are constant variations.

By the semi-simplicity of the category of variations of pure Hodge structures of fixed weight ([3]), we deduce that the local system $R^p g_* \mathbf{C}_{\tilde{Y}_U}$ has a subsystem which is isomorphic to a trivial local system with fiber $\mathrm{Im}(\bar{c}'_{p,Y})$ for $\bar{c}'_{p,Y} : H^p(\mathbf{C}_{Y \cap D'}) \to H^p(\mathbf{C}_{Y \cap D \cap D'})$). Then by the invariant cycle theorem ([3]), we deduce

$$\begin{aligned} &\mathrm{Im}(\bar{c}'_{p,Y}) \subset \mathrm{Im}(H^0(U, R^p g_* \mathbf{C}_{\tilde{Y}_U}) \to H^p(\mathbf{C}_{Y \cap D \cap D'})) \\ &= \mathrm{Im}(H^p(\mathbf{C}_{\tilde{Y}}) \to H^p(\mathbf{C}_{Y \cap D \cap D'})) \end{aligned}$$

where the second and third homomorphisms are derived from the restrictions to the fiber D' of g. Since the combinatorics of the closed strata are the same for $\tilde{X}$ and $D \cap D'$, we have

$$\mathrm{Im}(\bar{c}'_p) = \mathrm{Im}(H^p(\mathbf{C}_{\tilde{X}}) \to H^p(\mathbf{C}_{D \cap D'})).$$

Since $H^p(\mathcal{O}_{\tilde{X}}) \cong H^p(\mathcal{O}_X)$, we have our assertion.

(b) There is an obvious inclusion $\mathrm{Ker}(d_p) \subset \mathrm{Ker}(e'_p)$. We know already that

$$H^p(\mathcal{O}_X(-D - D')) \to H^p(\mathcal{O}_X(-D))$$

is surjective, because it is a statement which is independent of D. Thus we have $\mathrm{Ker}(d_p) = \mathrm{Ker}(e_p)$. Therefore it is sufficient to prove

$$\mathrm{rank}(e_p) = \mathrm{rank}(e'_p).$$

As explained in the next Step 6, the sheaves $R^p g_* \mathcal{O}_{\tilde{X}}$ are locally free for all p. Therefore we have $h^p(\mathcal{O}_D) = h^p(\mathcal{O}_{D'})$, where we denote $h^p = \dim H^p$. We compare two exact sequences

$$0 \to \mathcal{O}_X(-D) \to \mathcal{O}_X \to \mathcal{O}_D \to 0$$
$$0 \to \mathcal{O}_X(-D') \to \mathcal{O}_X \to \mathcal{O}_{D'} \to 0.$$

Since the corresponding h^p's are equal, we deduce that the ranks of the corresponding homomorphisms of the long exact sequences are equal, and we are done.

.

Step 6. This is our Theorem 15.

Step 7. Proof of (1) is the same as loc. cit. □

Finally we can easily generalize the vanishing theorem to the case of **Q**-divisors by using the covering lemma (Lemma 3):

Theorem 18. *Let X be a projective GSNC variety, S a normal projective variety, $f : X \to S$ a projective surjective morphism, and let L be a permissible* **Q***-Cartier divisor on X such that $m_1 L$ is a Cartier divisor and $\mathcal{O}_X(m_1 L)$ is generated by global sections for a positive integer m_1. Assume that the support of L is a GSNC divisor, $\mathcal{O}_X(m_2 L) \cong f^*\mathcal{O}_Z(L_S)$ for a positive integer m_2 and an ample Cartier divisor L_S on S, and for each closed stratum Y of X, the restricted morphism $f|_Y : Y \to S$ is surjective. Then the following hold:*

(1) Let $s \in H^0(X, \mathcal{O}_X(nL))$ be a non-zero section for some positive integer n such that the corresponding Cartier divisor D is permissible. Then the natural homomorphisms given by s

$$H^p(X, \mathcal{O}_X(K_X + \ulcorner L \urcorner)) \to H^p(X, \mathcal{O}_X(K_X + \ulcorner L \urcorner + D))$$

are injective for all p.

(2) $H^q(S, R^p f_* \mathcal{O}_X(K_X + \ulcorner L \urcorner)) = 0$ for $p \geq 0$ and $q > 0$.

Proof. We take a finite Kummer covering $\pi : X' \to X$ from another GSNC variety such that $\pi^* L$ becomes a Cartier divisor. Let G be the Galois group. Then we have

$$(\pi_* \mathcal{O}_X(K_X + L))^G \cong \mathcal{O}_X(K_X + \ulcorner L \urcorner).$$

Therefore our assertion is reduced to the case where L is a Caritier divisor. □

References

[1] Abramovich, D.; Karu, K. *Weak semistable reduction in characteristic 0.* Invent. Math. **139** (2000), no. 2, 241–273.

[2] Deligne, Pierre. *Theoreme de Lefschetz et criteres de degenerescence de suites spectrales.* (French) Inst. Hautes Études Sci. Publ. Math. No. 35 (1968), 259–278.

[3] Deligne, Pierre. *Théorie de Hodge. II.* (French) Inst. Hautes Études Sci. Publ. Math. No. 40 (1971), 5–57.

[4] Deligne, Pierre. *Théorie de Hodge III.* (French) Inst. Hautes Études Sci. Publ. Math. No. 44 (1974), 5–77.

[5] Du Bois, Philippe. *Complexe de de Rham filtre d'une variete singuliere.* Bull. Soc. Math. France 109 (1981), no. 1, 41–81.

[6] Fujino, Osamu. *On Kawamata's theorem.* in Classification of Algebraic Varieties, European Math. Soc., 2011, 305–315.

[7] Hironaka, Heisuke. *Resolution of singularities of an algebraic variety over a field of characteristic zero. I, II.* Ann. of Math. (2) **79** (1964), 109–203; ibid. (2) **79**(1964), 205–326.

[8] Kawamata, Yujiro. *Characterization of abelian varieties.* Compositio Math. **43** (1981), 253–276.

[9] Kawamata, Yujiro. *Pluricanonical systems on minimal algebraic varieties.* Invent. Math. **79** (1985), 567–588.

[10] Kawamata, Yujiro. *Semipositivity theorem for reducible algebraic fiber spaces.* arXiv:0911.1670 math.AG.

[11] Kawamata, Yujiro. *Variation of mixed Hodge structures and the positivity for algebraic fiber spaces.* arXiv:1008.1489 math.AG.

[12] Kollár, János. *Higher direct images of dualizing sheaves. I.* Ann. of Math. (2) 123 (1986), no. 1, 11–42.

[13] Kollár, János. *Higher direct images of dualizing sheaves. II.* Ann. of Math. (2) 124 (1986), no. 1, 171–202.

[14] Nakayama, Noboru. *Hodge filtrations and the higher direct images of canonical sheaves.* Invent. Math. 85 (1986), no. 1, 217–221.

[15] Schmid, Wilfried. *Variation of Hodge structure: the singularities of the period mapping.* Invent. Math. 22 (1973), 211–319.

Department of Mathematical Sciences, University of Tokyo,
Komaba, Meguro, Tokyo, 153-8914, Japan
kawamata@ms.u-tokyo.ac.jp